$$k_n = -\frac{\pi^2}{\xi^2} n^2 - 2\left(\frac{\lambda^2}{1+\xi\lambda} + \frac{\varkappa}{\xi}\right)[1 + o(1)], \qquad \varphi'(k_n) = -\frac{\pi^4}{2\varkappa\xi^3} n^4 [1 + o(1)] \qquad (4.4)$$

Equations (4.4) show that for $t > 0$ the series (4.1)–(4.3) can be differentiated term by term an innumberable number of times, for $t = 0$ once. All derivatives with respect to $\omega_0(t)$ starting with the second tend to infinity as $t \to 0$. Separating the principal terms for $t \to \infty$ in series (4.1), (4.2) and (4.3) we establish the following. If roots k_{01} and k_{02} are outside of the interval $(\zeta_1, 0)$, the sphere passes through the equilibrium position odd and finite number of times, or an infinite number of times. If roots k_{01} and k_{02} belong to the interval $(\zeta_1, 0)$, then the sphere does not pass through the position of equilibrium but only approaches it monotonically and indefinitely with an angular velocity which does not become zero for $t > 0$.

We note that this derivation applies also to the analogous problems of small rotational or longitudinal oscillations of an elastically coupled rigid plane in a viscous fluid, bounded by stationary walls which are parallel to the oscillating plane [3], or of an infinite cylinder in a fluid bounded by a stationary coaxial cylindrical wall. The analysis for the infinite cylinder does not differ substantially from the one carried out in this paper and leads to the same basic conclusions.

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Translated by B. D.

COMPARISON OF RESULTS OF AN ANALYSIS OF TRANSIENT WAVES IN SHELLS AND PLATES BY ELASTICITY THEORY AND APPROXIMATE THEORIES (*)

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Transient strain wave propagation in elastic shells and plates caused by an effect (application of loading, communication of displacements or velocities) which grows to a maximum or exerts influence in a time interval less than the time of strain wave traversal of a path equal to the characteristic dimension of the middle surface is considered on the

*) Material of a paper expounded by the author in two reports to the Third All-Union Congress of Theoretical and Applied Mechanics (Moscow, Jan.-Feb., 1968), and summarized in a report to the XIIth International Congress of Applied Mechanics (Standford, August, 1968).

basis of linear theory. A brief survey is given of results obtained recently, and domains for a well-founded and effective applicability of methods of integrating the linear elasticity theory equations, approximate equations and methods of integration and methods of integrating approximate equations are exposed. These domains depend on the type of effect, and on the law of time variation of the effect.

1. Introduction. Let us consider transient shell and plate strains within the scope of linear elasticity theory.

According to elasticity theory the domain of application of an effect emerges as the source of primary elementary waves, which being reflected from the side surfaces, result in the appearance of a complex system of elementary waves. A complete, separate description of the rapidly increasing quantity of elementary waves is impossible in practice. Hence, two aspects corresponding to the following goals are being developed in theoretical investigations: (a) the determination of the location and intensity of elementary wave fronts, and (b) the approximation of the total contribution of elementary waves by the smoothing of all or almost all the elementary wave fronts. Among the latter are all methods of analyzing transients based on approximate plate and shell theories.

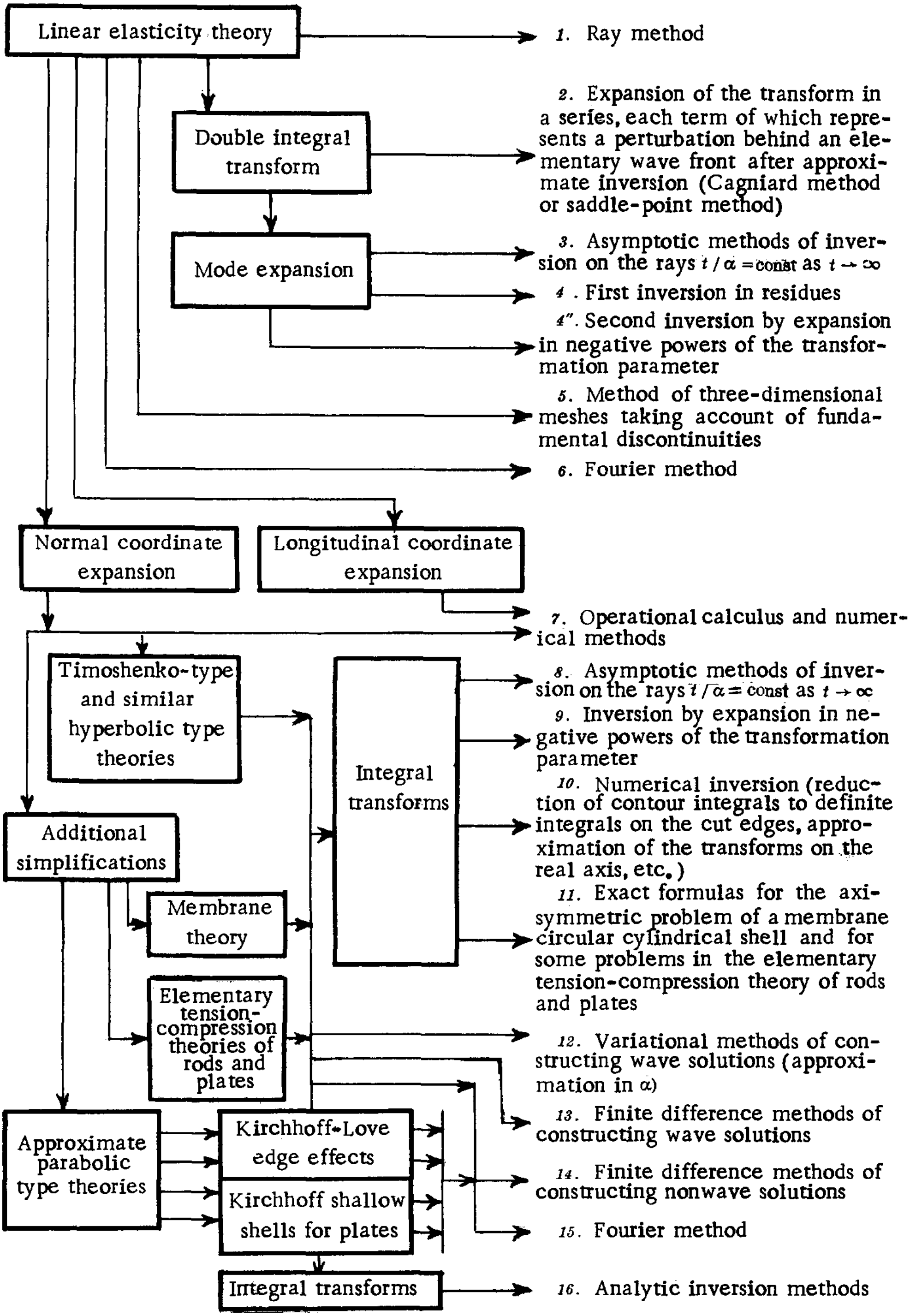
An attempt is made below to utilize existing results to clarify the domains of well-founded and effective applicability of methods of integrating the elasticity theory equations, approximate equations, and methods of integrating the approximate equations. The main attention is hence paid to axisymmetric problems of the deformation of shells of revolution without singularities, as well as to axisymmetric and plane strain problems of plates under various local effects. Some remarks on other kinds of effects are presented at the end of the paper.

In posing the question of the applicability of approximate methods it is necessary to agree on which quantities to evaluate (norm of the accuracy estimate). It is assumed herein that displacements and their first derivatives, or quantities evaluated in terms of them without differentiation (stresses, say), will be desired quantities.

2. Methods of analysis. The existence of recently published survey papers [1 and 2] permits elimination of a detailed description of the methods. Hence, we shall utilize the appended diagram, and explanations wherein are listed all, insofar as the author knows, published and applied methods; we present an interpretation of Diagram, and also mention some subsequent results.

Let us note that one designation on the diagram actually combines an aggregate of methods which are similar in idea. Specific results of transient analysis in the literature refer to shells of revolution and plates, which certainly indicates the preeminent value of these objects in engineering. Meanwhile, the great mathematical difficulties originating in the case of objects of more complex configuration, should be noted. Primarily axisymmetric and plane problems are investigated, for which the time t, a coordinate α on the middle surface, and the normal coordinate z will be the independent variables. Moreover, some problems for which a sinusoidal dependence of the desired quantities on the other coordinate β on the middle surface is given, will be examined

Diagram of the methods



Aizenberg [15] investigated resonance phenomena in a cylindrical shell subjected to an acoustic pressure wave incident at an angle to the axis by reducing the problem to the analysis of cyclically symmetric problems. The axisymmetric deformation of two-layer cylindrical and spherical shells subjected to a smoothly distributed load has been examined by Rakhmatulin et al. [16], and a single-layer spherical shell in [17 and 18].

The methods *1—7* (see Diagram) have been applied to the approximate integration of the elasticity theory equations in specific problems.

Methods *1*, *2* are geared to a separate analysis of the elementary waves, and are utilized primarily in the case of plane layers in theoretical seismology. They permit a practical determination of the location and intensity of elementary wave fronts, as well as obtaining some information on the states of stress in narrow zones near the fronts. Moreover, method *2* is a sufficiently effective means of analyzing dynamic compliance in the domain of application of a local effect. Basic investigations on elementary wave formation in layers were performed in the thirties by Smirnov and Sobolev [19 and 20], and Vekua [21]. Petrashen' [6 and 7] and V. M. Babich (*) performed great services in developing methods *1* and *2*. Very interesting results are presented by Rosenfeld and Miklowitz [8], as well as in earlier papers [22–31].

The methods *3* and *4* are based on a preliminary decomposition of the wave process into modes. Hence, extensive information accumulated in the literature on the study of dispersion relations is utilized in applying them (see the survey papers [1, 4 and 32]). As a rule, however, the application of methods *3* and *4* demands an additional study of the modes. Method *3* actually combines a complex of asymptotic methods of inverting contour integrals on the rays $t/\alpha =$ const as $t \to \infty$. These methods have been utilized in problems to analyze rods [33–38] and plates [39–42], and also, recently, in some problems to analyze cylindrical shells, including the papers of Aizenberg and Slepian [14] and Aizenberg [15]. The method *4* was proposed in [8] for plates in order to clarify the contribution of the individual modes to the formation of the fundamental discontinuities, and to analyze the state of stress near them for large times. Consequently, possibilities have been established for approximating discontinuities by using approximate theories of plates.

It should be pointed out that the application of methods *2—4* is possible in those problems in which one succeeds in finding a suitable kernel of the integral transform with respect to the coordinate α. In view of this the double integral transforms until the present were used in case of bars, plates, cylindrical shells and spherical shells. Until recently a methodology existed of its application only for a definite type of mixed boundary conditions. Recently in a paper Miklowitz [43] suggested a methodology for cases in which either displacements or stresses are prescribed in a cross-section. Thus the range of applicability of the methods *2—4* is increased. In view of the large efficiency of the method *3*, attention should be paid to the development of a method of direct construction of solutions along the rays $t/\alpha =$ const with asymptotes $t \to \infty$ which are not based on the application of integral transforms and is applicable in the analysis of axially and cyclically symmetric deformation of shells of revolution of arbitrary shape without singular points. In addition to seeking such solutions by the WKB method, the idea of

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*) Geometric optic methods in the theory of nonstationary waves and fundamental solutions of hyperbolic equations. Doctoral Dissertation, Novosibirsk, 1961.

averaging deserves attention here, which is used, for instance, in papers [44 and 45] in the case of quasi-linear equations of second order.

The methods *1*, *2* permit practically to determine only the intensity of fronts of individual elementary waves, while methods *3*, *4* are efficient for large t ; therefore a summary approximation of the wave process during the initial stage of motion is of great interest with the aid of finite-difference and variational methods of integrating the equations of the theory of elasticity which guarantee the conservation of essential discontinuities of the solution.

Among the finite-difference methods until now the method of three-dimensional meshes *5*, has been factually used, which takes into account the most essential discontinuities of the solution which are exhibited analytically and which is using recurrence formulas for the solution of systems of algebraic equations arising from boundary conditions. This method was advanced for plates in [9] and was later used in the case of cylindrical shells in [10–12]. The finite-difference method of integrating the equations of elasticity theory has also been applied recently to the case of a finite cylindrical bar in [46].

The introduction of methods utilizing variational principles, particularly the principle proposed by Ainola [47], into numerical computations merits attention.

The Fourier method *6* has been utilized in [17 and 18] devoted to an analysis of the reflection of a sinusoidal signal, and acoustic wave in an ambient medium, from a spherical shell. However, as will be indicated at the end of the present paper, practically identical results can be obtained by the Fourier method based on approximate theories (method *15*) in the case of a signal wavelength substantially greater than the shell thickness.

In the case of a smoothly distributed effect, it is sufficient to take account of wave propagation in the direction of one coordinate in applied problems, and it is expedient to utilize standing-wave expansions in the other coordinate (method *7*). Wave propagation over the thickness of two-layer cylindrical and spherical shells was investigated by this means, for example, by Rakhmatulin et al. [16] by using the method of characteristics to solve the two-dimensional problem. An analogous idea was used by Novozhilov and Utesheva [48] in studying the dynamic torsion of a semi-infinite cylinder by using an expansion in the radial coordinate and subsequent analysis of the wave propagation relative to the longitudinal coordinate by operational calculus methods.

Approximate theories constructed by reducing the three-dimensional problem to two dimensions by using various methods [1 and 49] of representing the desired functions as series in given functions of z occupy an important place in the analysis of transients. The purpose of applying approximate theories in specific problems is to calculate approximately the coefficients of a certain quantity of the first members of the series as functions of t, α, β. Success in the application of such an approach depends on how well the series members taken into account approximate the wave process in the considered range of variation of t, α, β. The theories of Timoshenko-type, and the similar approximate theories in which zero and first order Legendre polynomials will be the given functions of z, quite justifiably became very popular.

The fundamental modifications of the utilized approximate theories and methods of integrating the approximate equations are listed in the diagram; a survey of the appropriate literature to 1966 is presented in [1 and 2]. Let us note some results of recent work.

Some development has been observed of methods to study wave processes of circular cylindrical shell deformation, which vary along one coordinate according to a sinusoidal law. In particular, Kolodiazhnyi and Filippov [50] investigated the influence of an oriented pressure wave in an arbitrary manner by using a triple integral transform (in t, α, β), which substantially reduces to a Fourier series expansion of the wave process in the longitudinal coordinate α and a subsequent asymptotic inversion of double contour integrals. Specific results have been obtained on the basis of shallow shell theory (method *16* . A similar problem has also been examined recently in [51]. Regrettably, it is impossible to ascertain the development of methods of analyzing wave processes for which the perturbed domain extends over both coordinates α, β of the middle surface.

Mainly recently the examination of axisymmetric and plane problems was still continued. Extensive application of finite-difference methods *13* within the scope of Timoshenko-type theories has been observed. The method of two-dimensional meshes of the type *13*, which conserves the discontinuities found by the method *9*, utilized earlier to compute plates [52], circular cylindrical shells [11] and spherical shells [53 and 54], has recently been extended to arbitrary shells of revolution; an analysis of discontinuities in an arbitrary shell of revolution has been made in [55], and the computation of conical and toroidal shells in [55 and 56]. Results by this method and the method *8* have been compared for plates in [57]. The method of characteristics [58] has also been applied in the case of a cylindrical shell within the span of Timoshenko-type theory.

Investigations on the basis of simpler computational models have continued. In particular, by using the Laplace transform Malyshev [59] obtained closed membrane solutions (the method *11*) for semi-infinite and finite circular cylindrical shells subjected to axisymmetric endface loading (approximate solutions constructed by using approximate inversion methods have earlier been known for these problems). Vardanian and Sarkisian [60] have examined the impact on an anisotropic plate and cylindrical shell. The solutions for these objects have been constructed, respectively, by the Fourier method *15* in double trigonometric series on the basis of the Kirchhoff and Kirchhoff-Love theories with the series coefficients evaluated by using a Laplace time transform. The contact conditions have here been formulated according to Hertz theory, and the transforms of the Fourier series coefficients have been found as expansions in powers of a small parameter characterizing the anisotropy (consequently, the influence of the anisotropy is manifested quite clearly).

Let us also note the development and application of new modifications of the variational methods *12*. In particular, the method of expansion in terms of α in an expanding interval [61–63] has been utilized widely in the case of plates. A new variational principle has been proposed in [64] for the analysis of shell deformation transients.

From the viewpoint of engineering applications, the tendency to avoid taking account of the influence of deformation of the structure flanging the shell in theoretical investigations should be acknowledged as a definite disadvantage. Wave reflection from the contour is not considered in the majority of cases; or very idealized boundary conditions, which are also typical in case the Fourier method *15* is used, are prescribed on the contour. The report of Senitskii [65] devoted to axisymmetric dynamic problems of computing a spherical shell with elastic fixing of the edges should be welcomed in such a background, although it has been performed in a non-wave formulation on the basis of the equations of shallow shell theory. Transverse forces at the edge are considered to be proportional to the normal displacements, and the bending moments to the angles of rotation. By utilizing a Laplace time transform, and a finite Hankel transform in the radial coordinate (method 16), a general solution is obtained on whose basis specific examples are examined (vibrations under the effect of stationary and pulse loadings).

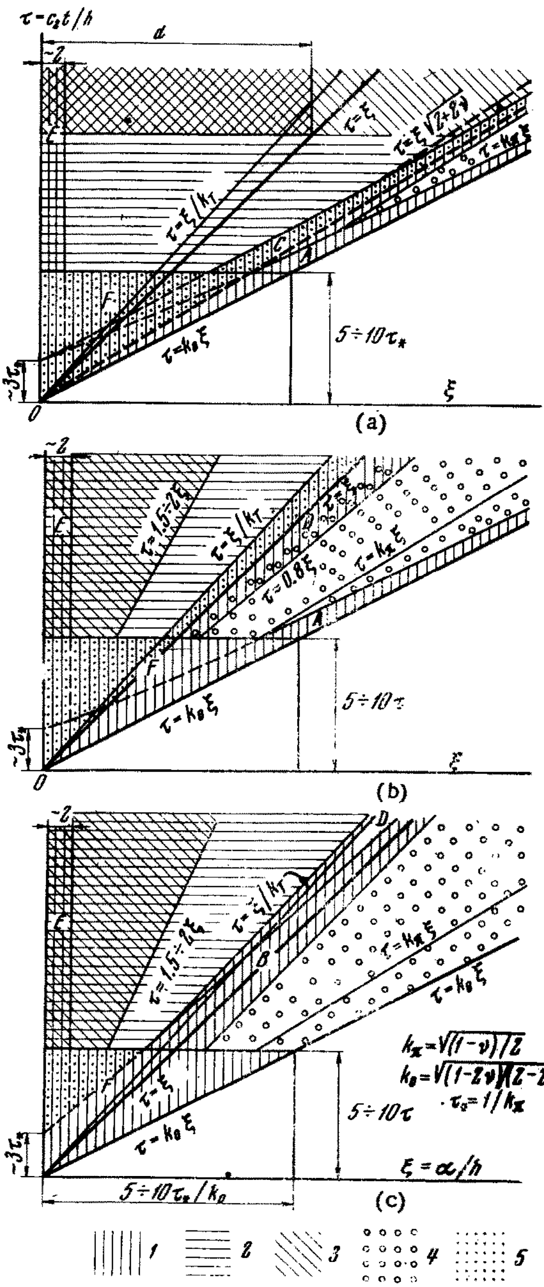

Fig. 1

3. Domains of effective applicability of the methods for axisymmetric and plane deformations caused by a local loading. Let us consider the axisymmetric deformations of a shell of revolution, as well as the axisymmetric and plane deformations of plates dependent on the coordinate α on the middle surface, and the normal coordinate z and the time t. Let E be the elastic modulus, ν the Poisson coefficient, c_1, c_2, c_R the propagation velocities of the compression-tension, shear, and Rayleigh waves, $2h$ the thickness of the shell or plate, $\xi = \alpha/h$, $\zeta = z/h$ are dimensionless coordinates, $A = h$, $B(\xi)$ the Lamé parameters of the middle surface, R_1, R_2 the middle surface radii of curvature, u, w the dimensionless (divided by h) displacements, σ_{ij} $(i,j = 1, 2, 3)$ the dimensionless (divided by $E/(1+\nu)$) stresses; $\tau = c_2 t/h$ the dimensionless time, d the width of the simple edge effect zone, exactly the same as in statics. Let us also utilize the notation

$$k_0 = \frac{c_2}{c_1}, \quad k_R = \frac{c_R}{c_2}, \quad k_\pi = \sqrt{(1-\nu)/2}, \quad \tau_* = \frac{1}{k_\pi},$$

$$\frac{\partial(\ldots)}{\partial\tau} = (\ldots)^{\cdot}, \quad \frac{\partial(\ldots)}{\partial\xi} = (\ldots)' \quad (3.1)$$

Speaking of approximate theories, let us use the concept of the following dimensionless quantities: U, W average longitudinal and normal displacements, ψ is the angle of rotation of the normal, T the longitudinal stress resultant, M the longitudinal bending moment, Q the transverse force. In a Timoshenko-type theory we assume that the shear coefficient k_T equals k_R.

To be specific, we shall have in mind effects of the type indicated in Table 1 in the section $\xi = 0$, where $f_0(\zeta), f_1(\zeta)$ denote functions satisfying the conditions

$$\int_{-1}^{+1} f_0(\zeta)\,d\zeta = 1, \quad \left|\frac{\partial f_0(\zeta)}{\partial \zeta}\right| \leqslant 1, \quad \int_{-1}^{+1} f_1(\zeta)\,d\zeta = 0, \quad \int_{-1}^{+1} \zeta f_1(\zeta)\,d\zeta = 0 \qquad (3.2)$$

The time dependence of the effect is defined in Table 1 in terms of $g_0(\tau)$ or $g_1(\tau)$ for each kind of effect (by either the upper or lower line). If the effect is given in displacements in terms of $g_0(\tau)$, we then use the notation

$$g_0^{\cdot}(\tau) = g_1(\tau) \qquad (3.3)$$

In conformity with the initial assumptions of linear elasticity theory, let us require continuity of $g_0(\tau)$, and let us impose the following condition on $g_1(\tau)$

$$|g_1(\tau)| \leqslant 1 \qquad (3.4)$$

Hence $g_1(\tau)$ can have finite discontinuities. Let us introduce the concept of the index of variability of the effect $\lambda(\tau)$ in the form

$$\lambda(\tau) = \frac{|g_1^{\cdot}(\tau)|}{\max|g_1(\tau)|} \qquad (0 \leqslant \tau \leqslant \tau_0) \qquad (3.5)$$

where τ_0 denotes the duration of the effect.

Types of effects

LT- longitudinal, tangential type;
LM- longitudinal, bending type;
NW- normal in displacements or velocities;
NQ- normal, of transverse force type;
LC- longitudinal, self-equilibrated type;
NC- normal, self-equilibrated type.

Table 1

Type	Boundary conditions at $\xi=0$: in elasticity theory		in plate and shell theory		
LT	$u = g_0(\tau)$; $\{ u^{\cdot}, u'$; $\sigma_{11} = g_1(\tau)$	$w = 0$; $\sigma_{13} = 0$	$U = g_0(\tau)$; $\{ U^{\cdot}, U'$; $T = g_1(\tau)$	$\{ \Psi = 0$; $M = 0$	$W = 0$; $Q = 0$
LM	$u = \zeta g_0(\tau)$; $\{ u^{\cdot}, u'$; $\sigma_{11} = \zeta g_1(\tau)$	$w = 0$; $\sigma_{13} = 0$	$U = 0$; $T = 0$	$\Psi = g_0(\tau)$; $\{ \Psi^{\cdot}, \Psi'$; $M = g_1(\tau)$	$W = 0$; $Q = 0$
NW	$u = 0$; $\sigma_{11} = 0$	$w = g_0(\tau)$; $w^{\cdot} = g_1(\tau)$	$U = 0$; $T = 0$	$\Psi = 0$; $M = 0$	$W = g_0(\tau)$; $W^{\cdot} = g_1(\tau)$
NQ	$u = 0$; $\sigma_{11} = 0$	$\sigma_{13} = f_0(\zeta)\,g_1(\tau)$	$U = 0$; $T = 0$	$\Psi = 0$; $M = 0$	$Q = g_1(\tau)$
LC	$\sigma_{11} = f_1(\zeta)\,g_1(\tau)$	$\sigma_{13} = 0$	—	—	—
NC	$\sigma_{11} = 0$	$\sigma_{13} = f_1(\zeta)\,g_1(\tau)$	—	—	—

For brevity, we have in mind wave propagation from the transverse $\xi = 0$ section in the direction of increasing ξ up to reflection from the $\xi = \xi_0$ edge. However, all the results expounded remain valid for wave propagation in the negative ξ direction as well.

Let us demand absence of singularities of the middle surface, and its smoothness in the following sense:

$$B \neq 0, \ | R_j' | \leqslant h \quad (j = 1, 2) \tag{3.6}$$

as well as the shell thinness

$$|R_j| \geqslant R_0, \ (j = 1, 2), \ h \,/\, R_0 = a \ll 1 \tag{3.7}$$

To the accuracy of elasticity theory, the problem is to integrate a system of two equations of motion which can be represented in our notation as

$$\begin{gathered}
\frac{\partial^2 u}{\partial \xi^2}\left(\frac{h}{H_1}\right)^2 + \frac{\partial u}{\partial \xi}\frac{h}{H_1}\left[\frac{\partial}{\partial \xi}\left(\frac{h}{H_1}\right) + \frac{1}{H_2}\frac{\partial B}{\partial \xi}\right] + \frac{\partial u}{\partial \zeta} k_0^2\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) + \\
+ \frac{\partial^2 u}{\partial \zeta^2} k_0^2 - \frac{\partial^2 u}{\partial \tau^2} k_0^2 + \frac{\partial^2 w}{\partial \xi \partial \zeta}\frac{h}{H_1}(1 - k_0^2) + \frac{\partial w}{\partial \xi}\frac{h}{H_1}\left[\frac{1 + k_0^2}{\zeta_1} + \frac{1 - k_0^2}{\zeta_2}\right] + \\
+ u\frac{h}{H_1}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{H_2}\frac{\partial B}{\partial \xi}\right) + \frac{h}{R_1} k_0^2\left(\frac{1}{\zeta_2} - \frac{1}{\zeta_1}\right)\right] + w\frac{h}{H_1}\frac{\partial}{\partial \xi}\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) = 0 \\
\frac{\partial^2 w}{\partial \xi^2}\left(\frac{h}{H_1}\right)^2 k_0^2 + \frac{\partial w}{\partial \xi}\frac{h}{H_1} k_0^2\left[\frac{\partial}{\partial \xi}\left(\frac{h}{H_1}\right) + \frac{1}{H_2}\frac{\partial B}{\partial \xi}\right] + \frac{\partial w}{\partial \zeta}\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) + \\
+ \frac{\partial^2 w}{\partial \zeta^2} - \frac{\partial^2 w}{\partial \tau^2} k_0^2 + \frac{\partial^2 u}{\partial \xi \partial \zeta}\frac{h}{H_1}(1 - k_0^2) + \frac{\partial u}{\partial \zeta}(1 - k_0^2)\frac{1}{H_2}\frac{\partial B}{\partial \xi} - \\
- \frac{\partial u}{\partial \xi}\frac{h}{H_1}\frac{1 + k_0^2}{\zeta_1} - u\left[\frac{1}{H_2}\left(\frac{k_0^2}{\zeta_1} + \frac{1}{\zeta_2}\right)\frac{\partial B}{\partial \xi} + k_0^2\frac{h}{H_1}\frac{\partial}{\partial \xi}\left(\frac{1}{\zeta_1}\right)\right] - w\left(\frac{1}{\zeta_1^2} + \frac{1}{\zeta_2^2}\right) = 0
\end{gathered} \tag{3.8}$$

where we have introduced the following notation:

$$H_1 = h\left(1 + \frac{\zeta h}{R_1}\right), \quad H_2 = B\left(1 + \frac{\zeta h}{R_2}\right), \quad \zeta_1 = \left(1 + \frac{\zeta h}{R_1}\right)\frac{R_1}{h}, \ \zeta_2 = \left(1 + \frac{\zeta h}{R_2}\right)\frac{R_2}{h} \tag{3.9}$$

Speaking of integrating system (3.8), we have in mind: (a) zero initial conditions, (b) the boundary conditions at $\xi = 0$ indicated in Table 1; (c) the boundary conditions $\sigma_{13} = 0$, $\sigma_{33} = 0$ at $\zeta = \pm 1$. In a Timoshenko-type theory and in Kirchhoff-Love theory the boundary conditions (b) take the form indicated in Table 1, and boundary conditions (c) are converted into the condition of no distributed loading.

Let us note that the location and amplitude of the fronts, as well as the rapidly varying states of stress near them can be analyzed by utilizing the simplified equations (3.8), in which terms containing the second and first derivatives are conserved. In a first approximation such an analysis can be carried out on the basis of still more simplified equations

$$\begin{gathered}
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial u}{\partial \xi}\frac{1}{B}\frac{\partial B}{\partial \xi} + \frac{\partial^2 u}{\partial \zeta^2} k_0^2 - \frac{\partial^2 u}{\partial \tau^2} k_0^2 + \frac{\partial^2 w}{\partial \xi \partial \zeta}(1 - k_0^2) + \\
+ \frac{\partial u}{\partial \zeta} k_0^2\left(\frac{h}{R_1} + \frac{h}{R_2}\right) + \frac{\partial w}{\partial \xi}\left[\frac{(1 + k_0^2)h}{R_1} + \frac{(1 - k_0^2)h}{R_2}\right] = 0 \\
\frac{\partial^2 w}{\partial \xi^2} k_0^2 + \frac{\partial w}{\partial \xi} k_0^2\frac{1}{B}\frac{\partial B}{\partial \xi} + \frac{\partial^2 w}{\partial \zeta^2} - \frac{\partial^2 w}{\partial \tau^2} k_0^2 + \frac{\partial^2 u}{\partial \xi \partial \zeta}(1 - k_0^2) + \\
+ \frac{\partial u}{\partial \zeta}(1 - k_0^2)\frac{1}{B}\frac{\partial B}{\partial \xi} + \frac{\partial w}{\partial \zeta}\left(\frac{h}{R_1} + \frac{h}{R_2}\right) - \frac{\partial u}{\partial \xi}(1 + k_0^2)\frac{h}{R_1} = 0
\end{gathered} \tag{3.10}$$

whose coefficients have an error of order a.

Only the coefficients of the last two members in (3.10) depend on R_j. Hence, for the effects indicated in Table 1, the amplitudes of the fundamental discontinuity and the state of stress near it are independent of R_j. For example, for effects of LT, LM type a discontinuity in the derivatives of u with respect to τ and ξ being displaced with the velocity c_1 turn out to be strongest. Hence, the discontinuity in u will be proportional to $\sqrt{B(0) \,/\, B(\xi)}$ with error on the order of a, and independent of R_j, while the discontinuities depend essentially on R_j. For effects of NW, NQ type, the discontinuity in derivatives of w with respect to τ and ξ being displaced at the velocity c_2 turn out to be strongest. Hence, with an error on the order of a the discontinuity in w will be proportional to $\sqrt{B(0)/B(\xi)}$ and independent of R_j, while the other discontinuities again depend essentially on R_j.

A theory of Timoshenko-type is based on approximations

$$u(\xi, \zeta; \tau) = U(\xi, \tau) + \zeta\Psi(\xi, \tau), \quad w(\xi, \zeta; \tau) = W(\xi, \tau) \tag{3.11}$$

Its equations of motion are constructed to different accuracy in part of the secondary terms. The following modification of the equations

$$L_{i1}U + L_{i2}\Psi + L_{i3}W = 0 \qquad (i = 1, 2, 3) \tag{3.12}$$

$$L_{11} = P, \quad L_{12} = L_{21} = \frac{Kh}{R_1} \qquad \left(K = k_T^2 \frac{1-\nu}{2}\right)$$

$$L_{13} = L_{31} = \left[\frac{(1+K)h}{R_1} + \frac{\nu h}{R_2}\right]\frac{\partial}{\partial\xi}$$

$$L_{22} = \frac{1}{3}P - K, \quad L_{23} = -K\frac{\partial}{\partial\xi}, \quad L_{32} = -K\left(\frac{\partial}{\partial\xi} + \frac{1}{B}\frac{\partial B}{\partial\xi}\right)$$

$$L_{33} = -K\frac{1}{B}\frac{\partial}{\partial\xi}B\frac{\partial}{\partial\xi} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\tau^2} + h^2\left(\frac{1}{R^2} + \frac{2\upsilon}{R_1R_2} + \frac{1}{R_2^2}\right)$$

$$P = \frac{1}{B}\frac{\partial}{\partial\xi}B\frac{\partial}{\partial\xi} + \frac{\nu}{B}\frac{\partial^2 B}{\partial\xi^2} - \left(\frac{1}{B}\frac{\partial B}{\partial\xi}\right)^2 - \frac{1-\nu}{2}\frac{\partial^2}{\partial\tau^2} \tag{3.13}$$

is justified in specifically investigated problems.

In some cases the following assumptions are admissible:

$$L_{12} = L_{21} = 0, \quad L_{32} = -K\frac{\partial}{\partial\xi}, \quad P = \frac{1}{B}\frac{\partial}{\partial\xi}B\frac{\partial}{\partial\xi} - \frac{1-\nu}{2}\frac{\partial^2}{\partial\tau^2} \tag{3.14}$$

To the accuracy of the system (3.12), discontinuities in the derivatives of U, Ψ, W determined approximately from the conditions $PU=0$, $P\Psi=0$, $L_{33}W=0$. will be fundamental, respectively, for effects of LT, LM, NW type. If B varies sufficiently slowly, then the functions found from these conditions are almost proportional to $\sqrt{B(0)/B(\xi)}$. In the first two cases, the strongest discontinuity is propagated with the velocity c_2/k_π, and in the third case with the velocity c_2k_T.

A more detailed analysis of the discontinuities and the fields near the fronts can be carried out by method $\mathcal{9}$. Let $V_i(s,\xi)$ $i=1,2,3$ be the Laplace transforms, respectively, of $U(\tau)\cdot\varphi(\tau,\xi)\cdot W(\tau,\xi)$. Then for R_1R_2B' varying sufficiently slowly, such an analysis reduces to constructing the asymptotic solution, as $s\to\infty$, of the Laplace transformed system (3.12) in the form

$$V_i(s,\xi) = \frac{1}{\sqrt{B(\xi)}}\sum_{j=1}^{3} C_{ij}(s,\xi)\,e^{-\lambda_i(s)\xi}, \quad \chi_j = k_js + \frac{g_j}{s} + \dots$$

$$k_1 = k_2 = \sqrt{(1-\upsilon)/2}, \quad k_3 = 1/k_\pi, \quad g_1 \sim a^2 \quad g_2 \sim -1, \quad g_3 \sim 1, \dots$$

Here $C_{ij}(s,\xi)$ are negative powers of s multiplied by a very slowly changing function of ξ. In the above-mentioned paper [55], first approximations $\lambda_j = k_js$ were utilized, whereupon incorrect expressions were obtained for some of the $C_{ij}(s,\xi)$

According to Timoshenko-type shell theory, two waves are propagated with the velocity c_2/k_π, and one with the velocity c_2k_T. According to membrane theory, one wave propagated at the velocity c_2/k_π exists. According to elasticity theory a quasifront is propagated with velocity c_2/k_π, and Rayleigh surface waves with the velocity c_2k_T.

Now, let us present deductions obtained as a result of complex application and comparison of various methods for different effects and objects. We hence have in mind the evaluation of the following quantities: u, u, u', w, w, w', σ_{11}, σ_{22}, σ_{13}.

On a large scale, we distinguish the following states of stress:

I – rapidly changing states of stress of wave type, not described by using approximate theories and characterized by the fact that the amplitudes of u, w will be small, and the amplitudes of some of their first derivatives will be commensurate with their maximum values throughout the whole process;

II – states of stress described by using the Timoshenko-type theory, or simpler approximate theories;

III – rapidly varying states of stress not described by utilizing approximate theories and characterized by the smallness of the amplitudes of u and w as well as of their first derivatives and σ_{ij}, as compared with the maximum amplitudes of these quantities throughout the whole process;

IV – quasi-static Saint Venant edge effects at the point of application of the effect.

States of stress of types I, II are of principal interest. Questions of the existence of these states of stress, and the selection of effective methods of constructing them as a function of the kind of effect, and the law of its time variation will be clarified by the use of Table 2 and Fig. 1, a–c.

Existence of the states of stress I, II, and also methods for constructing them

Table 2

Case	Time change of the effect		Type of effect	
			LT, LM, NW, NQ	LC, NC
1	smooth change	$\lambda \ll 1$ for $0 \leqslant \tau \leqslant \tau_0$, $\tau_0 \gg 1$	II; nonwave solution by methods *14 — 16*	
2	rapid loading	$\lambda \lesssim 1$ for $0 \leqslant \tau \leqslant \tau_0$ $\lambda = 0$ for $\tau > \tau_0$, $\tau_0 \gtrsim 1$	II; wave solution by methods *8 — 13*	
3	very rapid loading	$\lambda \gg 1$ for $0 \leqslant \tau \leqslant \tau_0$ $\lambda = 0$ for $\tau > \tau_0$, $0 \leqslant \tau_0 \ll 1$	I and II, respectively, in domains indicated for LT in Fig. 1a, for LM in Fig. 1b and for NW, NQ in Fig. 1c; here I is by methods *1—5* and *7*, II by methods *8 — 13*.	I; at the beginning of motion by methods *1, 2, 5, 7*; for large τ by methods *1, 3, 4, 7*
4	very brief effect	$\lambda \gg 1$ for $0 \leqslant \tau \leqslant \tau_0$ $g_1(\tau) = 0$ for $\tau > \tau_0$ $\tau_0 \ll 1$	I – analysis of discontinuities on elementary wave fronts and their rapidly changing states of stress by methods *1, 2, 3*	

Let us turn attention to the fact that method *6* does not figure in Table 2; it is ineffective in the case of local effect.

Case 1. States of stress I are absent. The state of stress IV will be most essential for effects of LC, NC type. The state of stress II, whose main components will be comparable to slowly varying functions of τ and ξ, will dominate for the LT, LM, NW, NQ effects. Hence, in practice it is acceptable to construct nonwave solutions by methods *14—16*. It is hence sufficient to apply the parabolic Kirchhoff-Love theory, of its simplified modifications, in the case of shells, and the Kirchhoff theory, or elementary compression-tension theory in the case of plates. In applying approximate modifications of Kirchhoff-Love theory one should be guided by the same principles as in corresponding static problems (also see the explanation of case 3 in the section on applicability of approximate theories).

Case 2. To some extent this case is analogous to the preceding case. However, the state of stress II has a more definite wave character for effects of LT, LM, NW, NQ type, and in certain ranges of variation of τ—ξ rapidly changing states of stress predominate, which require construction of a wave solution on the basis of a Timoshenko-type theory by methods *8—13*. The possibilities of replacing this theory by simpler ones, as well as the domains of effective applicability of the integrations methods are the same as in case 3.

Case 3. This case will be most complex, and we consider it separately for different kinds of effects. Its limiting case $\tau_0 = 0$ should be understood in such a way that

$$g_1(\tau) = CH(\tau), \quad C = \text{const} \tag{3.15}$$

In indicating the domain of existence of the state of stress I for $\tau \gg 1$ we assume that the amplitudes of this state do not become negligibly small because of the decrease in $\sqrt{B(0)/B(\xi)}$ (the situation is simplified in the opposite case, and becomes similar to case 2 as τ grows).

Let us consider the LT type of effect in the case of shells of revolution by utilizing Fig. 1a, in which the following notation is used: *1* – elasticity theory; *2* – Timoshenko-type theory; *3* – membrane theory; *4* – low-amplitude oscillations (state of stress III); *5* – Timoshenko-type theory just for the evaluation of U; A is the zone of the state of stress I behind the main front $\tau = k_0\xi$, C is the zone of the state of stress I on the main quasi-front $\tau = k_\pi \xi$, F is the zone of the state of stress at the beginning of the motion, E – the zone of existence of the state of stress IV.

According to elasticity theory, at the beginning of the motion (in the zone F) it is expedient to consider the solution as the sum of two components, the first of which satisfies boundary conditions for $\xi = 0$, and transfers the discontinuities to the fronts $\tau = k_0\xi$, $\tau = \xi$, but does not satisfy boundary conditions for $\zeta = \pm 1$, and the second, which assures satisfaction of boundary conditions at $\zeta = \pm 1$, is the sum of primary elementary waves resulting from conditional sources at the points $\xi = 0$, $\zeta = \pm 1$. The first component, which is constant or varying slowly in the thickness of the object, can easily be constructed exactly by analytic means, or with an error acceptable in practice. However, the domain where it exists without the second component, contracts rapidly as τ grows.

The location and intensity of the elementary wave fronts of the second component of the solution can be determined by methods *1, 2*. For a complete description of the second component of the solution (with the exception of some smoothing of the secondary continuities), the most effective of the methods introduced turns out to be method *5*, but the application of variational methods and the methods *7* apparently deserve attention. The volume of the computations by means of method *5* increases in proportion to τ^2.

Results of an analysis of specific problems, including results of a computation to $\tau / k_0 = 40$ published in [11 and 12], show that in addition to the main discontinuity at the beginning of the motion, especially large amplitudes of the state of stress I are manifest in the region of points of emergence of the first bow waves at the approximate ranges $j\tau_* / k_c$ $(j = 1, 2, 3)$ behind the main discontinuity along ξ. However, even for $\tau \approx 5 \div 10\ \tau_*$ the state of stress I is localized in narrow zones A and C, where a Timoshenko-type theory is applicable outside zone C (Fig. 1a). The above is also verified by comparing results by methods *3* and *8*. Methods *1* and *4* are effective in zone A, and method *3* in zone C (for very large τ these zones become narrow, and lose practical value). A Timoshenko-type theory is also applicable in the domain *5* in parts of U.

Upon applying a Timoshenko-type theory at the initial stage of the motion, the contribution of Ψ and W in the first equation of the system (3.12) is negligible, and U can be found from Eq. $PU = 0$ [11 and 56]. To the accuracy of this equation U is independent of R_j. If B varies sufficiently slowly, then

$$U \approx \sqrt{B(0)/B(\xi)}\, U_0 \tag{3.16}$$

where U_0 is the solution of Eq.

$$U_0'' - \frac{1-\nu}{2} U_0^{\cdot\cdot} = 0 \tag{3.17}$$

In the case of thick shells, an analogous situation continues to exist even for comparatively large τ, but elementary compression-tension theory of a bar of variable cross section yields a better approximation sufficiently far from the front $\tau = \xi / \sqrt{2 + 2\nu}$.

While W is small compared to U, it is admissible to omit it also from expressions for the tangential stress resultants of a Timoshenko-type theory. It is important to note here that the maximum amplitudes of W and ψ at the beginning of the motion will be on the order of the relative thickness less than the maximum amplitudes of U. However, the growth in W as compared with U proceeds more rapidly as τ grows, than does the shell thinness, and hence, in the case of thin shells the mentioned approximate methods yield an approximate representation of only the displacement U for $\tau \gg 1$. However, after the front $\tau = \xi$ has passed the zone of practical damping of the simple edge effects of width a dependent on the relative thickness, another method of simplification turns out to be effective: partition of the state of stress into a membrane state and simple dynamic edge effects (Fig. 1a). The membrane state predominates in parts of U and $\dot{T}$ in domains *2* and *3*. Let us note that the possibility of the mentioned partition was given a foundation analytically in 1961 by Alumiae [66], and is verified by an analysis of numerical results for different objects.

An LT type effect in the case of a plate causes a deformation which is symmetric relative to the middle surface, and a Timoshenko-type theory as well as membrane theory become equivalent to elementary compression-tension theory of plates, which replaces the above-mentioned approximate theories in their domains of applicability.

Let us now consider LM type effects by using Fig. 1b in which the notation is the following: *1* – elasticity theory, *2* – Timoshenko-type theory, *3* – Kirchhoff-Love theory (for shells), or Kirchhoff theory (for plates), *4* – low-amplitude oscillations (state of stress III), *5* – Timoshenko-type theory for the evaluation of W; A, D, F are zones of state of stress I, E the zone of existence of the state of stress IV.

From the viewpoint of the applicability and efficiency of the approximate methods of integration, the situation in zones F and A is the same as in the case of the LT type effect. For $\tau > 5 \div 10\ \tau_*$ a domain of very low amplitude oscillations exists beyond zone A. However, as the ratio τ / ξ increases, the amplitudes again increase gradually for any $\tau = $ const, and deserve practical attention at $\tau \approx 0{\div}8 \div 0{\div}9\ \xi$. In this connection, a domain D, where method *3* is effective, appears. With the exception of a rough calculation of W in domain *5*, the Timoshenko-type theory assures more-or-less reliable results only in domain *2*, where it can be replaced by Kirchhoff-Love theory for shells and by Kirchhoff theory for plates in domain *3*. At the beginning of the motion it is hence admissible to find W and Ψ from the last two equations of the system (3.13), by neglecting the influence of U and utilizing nontangential boundary conditions.

The coefficients of such a system of equations, which are equivalent to the corresponding equations of Kirchhoff-Love theory, are independent of R_j, and depend on the geometric parameters of the object only in terms of B. Hence, known numerical results for plane ($B = \text{const}$) and axisymmetric ($B = \xi$) deformations of plates permit prediction of the character of the displacements, bending moments, and transverse force in the case of different shells of revolution.

In particular, the mentioned quantities will be practically completely identical at the beginning of the motion for the plane deformation of a plane and the axisymmetric deformation of a circular cylindrical shell, and the deformation of shallow shells differs slightly from the axisymmetric deformation of plates. As τ grows, the influence of R_j and U grows gradually in the case of shells, where the nontangential factors of the state of stress, which acquire the character of a simple edge effect, dominate in region *3*. Hence, an effective partition of the state of stress into a membrane state and simple edge effects is possible in domain *3* for $\tau \gg 1$.

Let us turn to an examination of NW, NQ type effects by using Fig. 1c in which the numbers 1, ..., 5 and the letters F, E are applicable in the same sense as in Fig. 1b. The very essential difference from the case of the LM type effect is that the principal discontinuity now takes place on the $\tau = \xi$ front, consequently, a state of stress I originates in zone B which can be investigated by methods *1*, *3*, *4*. Method *3* is effective in the domain D. A Timoshenko-type theory is applicable in the domain of Rayleigh surface waves behind the quasifront. In the rest of the domain the applicability of the approximate methods is the same as for the LM type effects.

Let us now consider the question of the efficiency of methods of integrating equations of a Timoshenko-type theory for LT, LM, NW, NQ effects. The remarks cited here refer equally to cases 2 and 3. It is known [1] that some modifications of method *10* (see Diagram), particularly the reduction of contour integrals to definite integrals on the lips of slits in the plane of the complex transformation parameter, have been applied in the first papers devoted to using a Timoshenko-type theory in problems to compute plates. Extension of these methods to shells does not turn out to be expedient because of the large number of branch points.

Finite difference methods *13*, including the modification of the two-dimensional mesh method in which the main discontinuities exposed by the method *9* as well as the method of characteristics are conserved, will be most effective at the beginning of the motion. When preliminary information exists on the wave process which will assure a successful selection of the approximating functions, the methods *12* are also very efficient. However, the volume of calculations grows approximately with τ^2 by methods *12*, *13*. Hence, the application of the method *9* in the region of the main discontinuity merits attention for $\tau \gg 1$, and application of the method *8* far from the main discontinuity. A comparison of results according to methods *8* and *13* indicates that the method *8*, realized in a first approximation, is applicable for plates and thick cylindrical shells for

$\tau \gg 10 \div 40$, where the lower limit of applicability depends essentially on the quantities under consideration.

Exact solutions (method *11* in Diagram), or solutions guaranteeing a good approximation in the whole range $\xi < \xi_0, \tau < k_0\xi_0$, exist for certain problems within the limits of the membrane theory, as well as the elementary theories of rods and plates. In the remaining cases, the methods *12*, *13* are again effective at the beginning of the motion, and methods *8*, *9* for large τ.

Case 4. Because of the smallness of the displacements, their displacements which have the greatest amplitudes in contracting zones (Fig. 1a) at the front $\tau = k_0\xi$ and the quasifront $\tau = k_\pi \xi$ under LT, LM, LC type effects, and in the contracting zone (Fig. 1b) at the front $\tau = \xi$ for NW, NQ, NC type effects, are of principal interest. Methods *1*, *2*, *4* should be applied at the fronts $\tau = k_0\xi$, $\tau = \xi$, and method *3* at the quasifront $\tau = k_\pi \xi$.

Finally, let us note that a shear coefficient $k_T = k_R$ was assumed in the Timoshenko-type theory in the discussions made since, as a rule, this will assure the best results of approximating the dispersion relations used in realizing the method *8*. However, some change in the numerical value of k_T does not change essentially the limits of the well-founded applicability of this theory.

4. Some generalizations. As before, let us speak of the evaluation of displacements, their first derivatives and the stresses. In the previous section it has been mentioned that effects which are smoothly distributed and nonequilibrated in the thickness of the shell or plate, will cause wave processes in the case

$$\lambda(\tau) \lesssim 1 \tag{4.1}$$

for which the principal components (in the sense of the magnitude of the amplitudes of the desired quantities) of the state of stress can be approximated well enough by using a Timoshenko-type theory or simpler approximate theories. Satisfaction of condition (4.1) substantially guarantees "smoothness" of the excited elementary wave fronts in some sense, which assures the possibility of approximating the dependence of the principal stress states on ζ by using smooth functions of ζ introduced into the considerations in constructing the approximate theories of shells and plates. Hence, there is a foundation to assume that satisfaction of condition (4.1) will be necessary also for a correct description by using approximate theories of the displacements, their first derivatives, and the stresses in the case of a distributed effect.

For example, let a wave process be excited by a normal loading (σ_{33}) or a normal velocity ($w^{\cdot}$), whose distribution in ξ is given by the function $G(\xi)$ on the surface $\zeta = 1$ or $\zeta = -1$. Let us introduce into the analysis the index of variability of the effect in ξ by means of

$$r(\xi) = \frac{|G'(\xi)|}{\max|G(\xi)|} \qquad (0 \leqslant \xi \leqslant \xi_0) \tag{4.2}$$

If condition (4.1) and the condition

$$r(\xi) \lesssim 1 \tag{4.3}$$

are satisfied, then a Timoshenko-type theory yields a practically acceptable approximation of the wave process in axisymmetric problems for shells of revolution, as well as in axisymmetric and plane problems for plates. If conditions (4.1), (4.3) are satisfied in the strong sense $\lambda \ll 1$, $r \ll 1$, then as a rule, a Timoshenko-type theory can be replaced by simpler theories.

The mentioned deductions are in good agreement with deductions on the applicability of approximate theories of asymptotic estimates relative to the index of variability of the state of stress. Such asymptotic estimates have been obtained, for example, in [67–69] for dynamic problems of the computation of circular cylindrical shells.

Method *6* which was acknowledged as inefficient in the preceding Sect. for the case of local loading, can be efficient if conditions (4.1),(4.3) are satisfied in a sufficiently strong sense, and condition (3.7) is not satisfied. If $\lambda \ll 1$ and $r \ll 1$, then method *15* is most efficient in many cases. Hence, in particular, it follows that diagrams characterizing the echo of sinusoidal pulses from spherical shells $R_1 = R_2 = R_0$ in water and in air, obtained in [18, 70 and 71] on the basis of method *6*, can be found for sufficiently thin shells by using the method *15*, and with a great deal less difficulty; if the pulse wavelength exceeds the thickness of the spherical shell by a great deal, and $R_0/h \gg 10$, then it is sufficient to realize method *15* on the basis of membrane theory.

A criterion of type (4.1) can easily be generalized also to the case of a wave process dependent on two coordinates of the middle surface.

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DYNAMIC CHARACTERISTICS OF AN ELECTROMAGNETICALLY DRIVEN TRIGGER REGULATOR

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The dynamic characteristics of an electromagnetically driven trigger regulator with one pulse per period is considered for a model with one and a half degrees of freedom. The method of point transformations is used to find the decomposition of the parameter space into domains in each of which the system under investigation has the same qualitative structure of the decomposition of the phase space into trajectories. The parameter value ranges in which complex periodic motions occur in the system are established.

1. The dynamic model and equations of motion. The motion of electromechanical trigger regulators with electromagnetic drive [1 and 2] can be investigated with the aid of the following model. Oscillator *1* of soft-magnetic material (Fig. 1) oscillates under the action of the linear restoring force exerted by spring *2* and the pulses produced by interaction of the oscillator with pulse coil *3*. As the oscillator moves from left to right, pin *4* makes contact with leaf spring *5* near the position of static equilibrium, bends it, and closes the electrical circuit through contact *6*. During interaction of the magnetic field of pulse coil *3* with oscillator *1* (which is close to the coil at this instant), the latter received a mechanical impulse. As the oscillator continues to move forward (from left to right) leaf spring *5* slips out from under pin *4*, the circuit is broken, and the pulse ceases. The electrical circuit is not closed during the reverse motion of the oscillator.

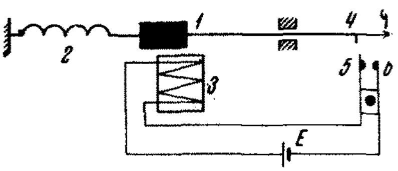

Fig. 1

We shall make the following simplifying assumptions.

1. There is no sparking when the electrical circuit is broken, and the circuit resistance, which is finite with the circuit closed, becomes infinite as soon as the circuit is opened [3].